

# Special values of Jacobi's first theta function

István Mező<sup>1</sup>

*Department of Applied Mathematics and Probability Theory, Faculty of  
Informatics, University of Debrecen, Hungary*

---

## Abstract

Based on R. W. Gosper's  $q$ -trigonometry and his conjectures, we give new formulae for some specific values of the Jacobi theta function of index one. The calculations strenghten Gosper's conjecture on his addition formulas.

*Key words:*  $q$ -trigonometry,  $q$ -Gamma function, Theta functions of Jacobi  
*1991 MSC:* 05A15

---

## 1 Introduction

We shall prove – among others – that the theta function

$$\vartheta_1(z, q) = \sum_{n=-\infty}^{+\infty} (-1)^{n-\frac{1}{2}} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)iz} \quad (1)$$

is connected to Gosper's  $q$ -sine function  $\sin_q$ :

$$i\vartheta_1(iz \ln q, q) = q^{-z^2} \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}} \sin_q(\pi z). \quad (2)$$

$\sin_q\left(\frac{\pi}{4}\right)$  can be evaluated easily, so for the special case  $z = 1/4$  we will get that

$$i\vartheta_1\left(\frac{1}{4}i \ln q, q\right) = \frac{(\sqrt{q}; q)_{\infty} (q; q)_{\infty}^2}{(q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}}. \quad (3)$$

---

*Email address:* mezo.istvan@inf.unideb.hu (István Mező).

*URL:*

<http://www.inf.unideb.hu/valseg/dolgozok/mezoistvan/mezoistvan.html>  
(István Mező).

<sup>1</sup> Present address: University of Debrecen, H-4010, Debrecen, P.O. Box 12, Hungary

Latter identity uses Gosper's conjectures on the addition formulas of the  $q$ -sine and  $q$ -cosine functions. Mathematica calculations show that the left and right hand sides really coincide thus this can be considered as a certificate of the conjectures. Gosper [2] defined the  $q$ -sine and  $q$ -cosine functions as

$$\sin_q(\pi z) := q^{(z-1/2)^2} \frac{(q^{2z}; q^2)_\infty (q^{2-2z}; q^2)_\infty}{(q; q^2)_\infty^2} \quad (0 < q < 1), \quad (4)$$

$$\cos_q(\pi z) := q^{z^2} \frac{(q^{1-2z}; q^2)_\infty (q^{2z+1}; q^2)_\infty}{(q; q^2)_\infty^2} \quad (0 < q < 1) \quad (5)$$

with  $(x; q)_\infty = (1-x)(1-qx)(1-q^2x) \cdots$ .

Gosper's conjecture is that these functions satisfy duplication formulas similar to the classical ones [2]:

$$\sin_q(2z) = S \sin_{q^2}(z) \cos_{q^2}(z), \quad (6)$$

$$\cos_q(2z) = \cos_{q^2}(z) - \sin_{q^2}(z). \quad (7)$$

Here

$$S \equiv S(q) = q^{-\frac{1}{4}} \frac{(q^2; q^4)_\infty^4}{(q; q^2)_\infty^2}. \quad (8)$$

We shall use these relations as true statements. For example, formula (3) seems to be true with any floating point precision. This strenghten the conjectures (6) and (7) of Gosper.

In our proofs we use the  $q$ -Gamma function [1]

$$\Gamma_q(z) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1-q)^{1-z} \quad (0 < q < 1). \quad (9)$$

There exists a simple connection between the  $q$ -sine function and  $\Gamma_q$  [2]:

$$\sin_q(\pi z) = q^{\frac{1}{4}} \Gamma_{q^2}^2\left(\frac{1}{2}\right) \frac{(q^2)^{\binom{z}{2}}}{\Gamma_{q^2}(z) \Gamma_{q^2}(1-z)}. \quad (10)$$

This generalizes Euler's reflection formula

$$\sin(\pi z) = \frac{\pi}{\Gamma(z) \Gamma(1-z)},$$

where  $\Gamma$  is the Euler gamma function.

Having these notions, we present a proof of (2).

## 2 The proof of (2)

The definition of  $\Gamma_q$  immediately gives that

$$\Gamma_{q^2} \left( \frac{1}{2} \log_q(qy) \right) \Gamma_{q^2} \left( \frac{1}{2} \log_q(q/y) \right) = \frac{(q^2; q^2)_\infty^2 (1 - q^2)}{(qy; q^2)_\infty (q/y; q^2)_\infty} = \frac{(q^2; q^2)_\infty^3 (1 - q^2)}{(q^2; q^2)_\infty (qy; q^2)_\infty (q/y; q^2)_\infty}.$$

The denominator can be rewritten via Jacobi's triple product identity [1, p. 15]:

$$(q^2; q^2)_\infty (qy; q^2)_\infty (q/y; q^2)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} y^n.$$

Thus we get that

$$\frac{(q^2; q^2)_\infty^3 (1 - q^2)}{\Gamma_{q^2} \left( \frac{1}{2} \log_q(qy) \right) \Gamma_{q^2} \left( \frac{1}{2} \log_q(q/y) \right)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} y^n.$$

Realize that if  $z = \frac{1}{2} \log_q(qy)$ , then  $1 - z = \frac{1}{2} \log_q(q/y)$  and  $y = q^{2z-1}$ . Hence

$$\frac{(q^2; q^2)_\infty^3 (1 - q^2)}{\Gamma_{q^2}(z) \Gamma_{q^2}(1 - z)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{n(2z-1)}.$$

The special value

$$\Gamma_{q^2} \left( \frac{1}{2} \right) = \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty} \sqrt{1 - q^2} \quad (11)$$

and equation (10) yield the remarkable summation formula for  $\sin_q$ :

$$\sin_q(\pi z) = q^{(z-\frac{1}{2})^2} \frac{(q^2; q^2)_\infty}{(q; q)_\infty^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{n(2z-1)}. \quad (12)$$

The sum on the RHS can be easily transformed to the theta function  $\vartheta_1$ :

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} q^{n(2z-1)} = q^{-\frac{1}{4}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} q^{2n(z-1)}. \quad (13)$$

On the other hand, definition (1) gives that

$$ie^{-iy} \vartheta_1(y, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{2niy}.$$

If we choose  $y = i(1 - z) \ln q$ , then

$$ie^{-iy} \vartheta_1(y, q) = iq^{1-z} \vartheta_1(i(1 - z) \ln q, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} q^{2n(z-1)}.$$

This and (12)-(13) gives that

$$\sin_q(\pi z) = q^{(z-\frac{1}{2})^2} \frac{(q^2; q^2)_\infty}{(q; q)_\infty^2} q^{-\frac{1}{4}} i q^{1-z} \vartheta_1(i(1-z) \ln q, q).$$

Then a straightforward rearrangement and substitution  $z = 1 - z$  give our desired formula (2). (We remark that  $\vartheta_1$  can be transformed to the other theta functions  $\vartheta_{2,3,4}$ , so we can also have formulae connecting  $\sin_q$  to these functions.)

### 3 Special values for $\sin_q$

In what follows we derive an expression for  $\sin_q(\pi/4)$  and  $\sin_q(\pi/8)$ . The first one (together with (2)) will imply (3) at once.

We have

$$\sin_{q^2} \left( \frac{\pi}{4} \right) = q^{\frac{1}{8}} \frac{(q; q^2)_\infty}{(q^2; q^4)_\infty^2}, \quad (14)$$

$$\sin_{q^2} \left( \frac{\pi}{8} \right) = \sqrt{\sin_q \left( \frac{\pi}{4} \right) \frac{\sqrt{S^2 + 4} - S}{2S}} \quad (15)$$

Latter identity shows that we can not wait for a simple closed form expression for  $\sin_q(\pi/2^n)$  (and  $\sin_q(k\pi/2^n)$  in general). However, (15) tends to  $\sqrt{\frac{\sqrt{2}}{4}(\sqrt{2} - 1)}$ , which is  $\sin(\pi/8)$  (this is true, because  $S \rightarrow 2$  if  $q \rightarrow 1-$ , see the duplication formulas).

In order to prove (14), we put  $z = \pi/4$  in (6) and (7). Since  $\sin_q(\pi/2) = 1$  and  $\cos_q(\pi/4) = 0$  (see the definitions), we easily have that

$$\sin_{q^2} \left( \frac{\pi}{4} \right) = \cos_{q^2} \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{S}}.$$

According to the definition of  $S$ , a simple algebraic manipulation proves (14).

As an interesting corollary, (14) and (10) (together with (11)) yield the curious product formula

$$\Gamma_{q^2} \left( \frac{1}{4} \right) \Gamma_{q^2} \left( \frac{3}{4} \right) = (1 - q^2) \frac{(q^2; q^2)_\infty^4 (q; q^2)_\infty^2}{(q; q)_\infty^2 (\sqrt{q}; q)_\infty}.$$

This is the  $q$ -analogue of the classical product

$$\Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{3}{4} \right) = \pi \sqrt{2}.$$

Finally, we deal with (15). We apply our duplication formulas for  $z = \pi/8$ :

$$\begin{aligned}\sin_q\left(\frac{2\pi}{8}\right) &= S \sin_{q^2}\left(\frac{\pi}{8}\right) \cos_{q^2}\left(\frac{\pi}{8}\right), \\ \cos_q\left(\frac{2\pi}{8}\right) &= \cos_{q^2}^2\left(\frac{\pi}{8}\right) - \sin_{q^2}^2\left(\frac{\pi}{8}\right).\end{aligned}$$

Since  $\cos_q(z) = \sin_q(\pi/2 - z)$ , we get that the left hand sides coincide, and  $\cos_{q^2}(\pi/8) = \sin_{q^2}(3\pi/8)$ . Thus

$$S \sin_{q^2}\left(\frac{\pi}{8}\right) \sin_{q^2}\left(\frac{3\pi}{8}\right) = \sin_{q^2}^2\left(\frac{3\pi}{8}\right) - \sin_{q^2}^2\left(\frac{\pi}{8}\right). \quad (16)$$

Since  $\sin_q(z) = \sin_q(\pi - z)$  ( $0 \leq z \leq \pi$ ), the duplication formula for  $\sin_q$  with  $z = 3\pi/8$  easily gives that

$$\sin_{q^2}\left(\frac{3\pi}{8}\right) = \frac{\sin_q\left(\frac{\pi}{4}\right)}{S \sin_{q^2}\left(\frac{\pi}{8}\right)}.$$

Substituting this into (16), we win

$$\sin_q\left(\frac{\pi}{4}\right) = \frac{\sin_q^2\left(\frac{\pi}{4}\right)}{S^2 \sin_{q^2}^2\left(\frac{\pi}{8}\right)} - \sin_{q^2}^2\left(\frac{\pi}{8}\right).$$

And this is just a quadratic equation in the “variable”  $x = \sin_{q^2}^2\left(\frac{\pi}{8}\right)$ . Since the value  $\sin_{q^2}\left(\frac{\pi}{8}\right)$  is surely positive, the only one solution will be exactly (15).

## References

- [1] G. Gasper and M. Rahman, *Basic Hypergeometric Series* (second edition), Cambridge University Press, 2004.
- [2] R. W. Gosper, *Experiments and Discoveries in q-Trigonometry* In Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics. (Editors: F. G. Garvan and M. E. H. Ismail). Dordrecht, Netherlands: Kluwer, pp. 79-105, 2001.